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Taylor Instability in a Stratified Flow*

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CONTENTS

ABSTRACT

The effect of a mean flow and density stratification on the instability of a density discontinuity is investigated. The flow is of the jet type between parallel walls with a piecewise linear velocity profile. The stratification is statically stable. The dominant instability occurs for large values of the perturbation wavenumber. The stratification and flow are generally stabilizing, but the effect of the stratification is found to be destabilizing with respect to small wavenumbers.

1. Introduction

We shall be interested in how the instability of a heavy fluid superposed on a lighter one (Taylor instability) may be modified by a mean flow and a mean stratification of density. We assume an inviscid, incompressible fluid, and by stability we mean stability with respect to small perturbations in the linearized theory. Since we are primarily interested in modifications which are possibly stabilizing, we choose a velocity profile of the jet type between parallel plates, which is of the stable type for inviscid fluids according to Rayleigh's criterion since it does not have a point of inflection.

To simplify the mathematical treatment of the problem we assume a piecewise linear velocity profile, i.e., we assume two layers, in each of which the profile is linear, with an unstable density discontinuity at the interface. Within each layer we assume a density stratification which is statically stable (positive Richardson number). Such a stratification could arise from the temperature distribution of a hot, heavy fluid supported by a cooler, light fluid.

2. Formulation of the Eigenvalue Problem

2.1 Linearization of the Basic Equations

We assume an inviscid, incompressible fluid in a constant gravity field. The basic equations are, therefore, the momentum equation

$$\frac{D\vec{v}}{Dt} + \frac{1}{\rho}\operatorname{grad} p + \vec{g} = 0$$

the incompressibility condition

$$\frac{D\rho}{Dt} = 0$$

and the continuity equation, which may now be written

$$\operatorname{div} \vec{v} = 0$$

The equations have been written in non-dimensional form by introducing reference quantities V and ρ_o for the velocity and density and by assuming that the velocity field is characterized by a length ℓ and that the variation of density is characterized by a length h. Astericks will be used to denote dimensional quantities. Thus, the dimensionless gravitational constant is $g = (\ell/V^2)\, g_*$ and $\vec{g}$ is $(0, 0, g)$. The quantities $\vec{v}$, ρ, and p are the dimensionless velocity, density, and pressure fields, and D/Dt is the material derivative.

The dimensionless equations may be linearized about a mean velocity $[U(z), 0, 0]$, a mean density $\bar{\rho}(z)$, and a mean pressure $\bar{p}(z)$ by superimposing small perturbations (denoted by primes) on the mean fields

$$\vec{v} = (U + u', v', w')$$

$$\rho = \bar{\rho} + \rho'$$

$$p = \bar{p} + p'$$

and neglecting terms which are of the second order in primed quantities. Since the independent variables are cyclic except z, we assume the primed quantities to be of the form

$$q' = q(z) \exp i(\alpha x + \beta y - \alpha c t)$$

Assuming the wavenumbers α and β to be real, we may regard c as the (possibly complex) eigenvalue. Stability then depends on the imaginary part of c. If $\mathrm{Im}\, c > 0$ the perturbation grows in time and the flow is unstable. Moreover, it can be shown that for inviscid fluids if c is an eigenvalue then c^* is also an eigenvalue.[1] Therefore, we may infer instability if $\mathrm{Im}\, c \neq 0$.

The linearized equations may be solved for w, and if we further make the Boussinesq approximation of neglecting the density variation except where it is multiplied by the gravitational constant g, we obtain

$$\frac{d^2 w}{dz^2} - \left\{ k^2 + \frac{U''}{U - c} - \frac{k^2}{\alpha^2} \frac{J}{(U-c)^2} \right\} w = 0 \qquad (2.1)$$

where k is the total wavenumber

$$k^2 = \alpha^2 + \beta^2$$

and J is the local Richardson number

$$J = \frac{g_* l^2}{V^2 d} \quad \frac{d\bar{\rho}}{dz} \Big/ \bar{\rho}$$

Thus, we have static stability if $J > 0$.

2.2 The Eigenvalue Equation

We consider a flow of the jet type between parallel plates in two layers with an unstable density discontinuity $\Delta\rho$ at the interface (Fig. 1)

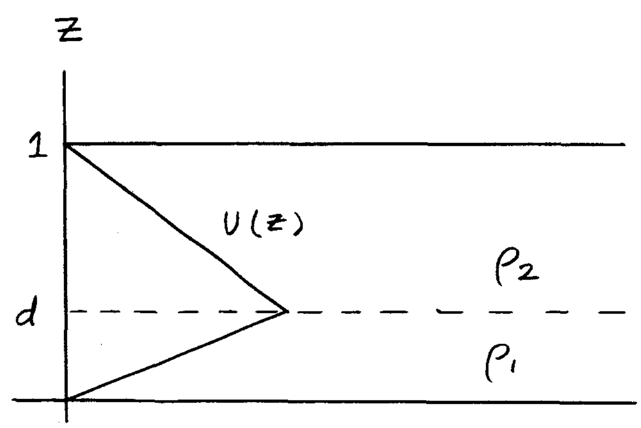

Fig. 1

The maximum velocity of the jet is used as the reference velocity V and the width of the channel as the reference length ℓ. Then the dimensionless velocity profile is

$$U = z/d \qquad 0 \le z \le d \qquad \text{(layer 1)}$$

$$= \frac{1-z}{1-d} \qquad d \le z \le 1 \qquad \text{(layer 2)}$$

where d is the (dimensionless) height of the interface and $\Delta\rho = \rho_2(d) - \rho_1(d) > 0$. Stratification within the layers is given by Richardson numbers $J_1(z)$ and $J_2(z)$.

The appropriate boundary conditions to impose are the vanishing of w at the walls $(z = 0, 1)$, and at the interface $(z = d)$ the continuity of w and the continuity of the pressure. In terms of w, the last condition is equivalent to the jump condition[2]

$$\Delta\left\{ \rho \left[U'w - (U-c)w' \right] \right\} = -\frac{K^2}{\alpha^2} \Delta\rho \, g \, \frac{w}{U-c}$$

where Δ denotes the jump in crossing the interface. Since g is dimensionless we should not think of it as the gravitational acceleration, but rather we may regard the quantity $\Delta\rho \, g$ as a sort of Richardson number associated with the density discontinuity.

Let $S = -\Delta U' = \dfrac{1}{d(1-d)}$

The function S is shown in Figure 2.

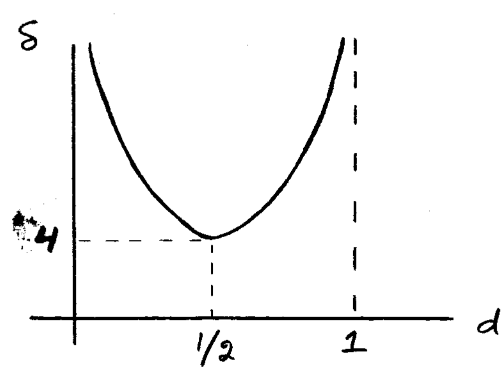

Fig. 2

If we let W_1 and W_2 be the solutions of eq. (2.1) for layer 1 and W_3 and W_4 the solutions for layer 2, by imposing the boundary conditions we obtain the eigenvalue equation for c :

$$\frac{W_2'(d)}{W_2(d)} \quad \frac{W_1'(d) / W_2'(d) - W_1(0) / W_2(0)}{W_1(d) / W_2(d) - W_1(0) / W_2(0)} - \Gamma$$

$$= \frac{W_4'(d)}{W_4(d)} \quad \frac{W_3'(d) / W_4'(d) - W_3(1) / W_4(1)}{W_3(d) / W_4(d) - W_3(1) / W_4(1)} \qquad (2.2)$$

where $\Gamma = \frac{1}{1-c} \left\{ \delta - \frac{\Delta\rho g / \cos^2\theta}{1-c} \right\}$

Theta is the angle which the wavenumber of the perturbation makes with the x-axis. Thus, $\alpha = k \cos\theta$.

If we assume that J_1 and J_2 are constants, explicit solutions of eq. (2.1) may be obtained in terms of Bessel functions. In the first layer ($0 \leq z \leq d$) eq. (2.1) may be written

$$\frac{d^2 W}{d\zeta^2} - \left\{ 1 - \frac{J_1 d^2 / \cos^2\theta}{\zeta^2} \right\} W = 0 \qquad (2.3)$$

where $\zeta = k\,(z - cd)$

Solutions of eq. (2.3), W_1 and W_2 , are of the form $\zeta^{1/2}\, Zn\,(i\zeta)$ where Zn is a Bessel function of order n and

$$n^2 = 1/4 - \frac{J_1 d^2}{\cos^2\theta}$$

In the second layer ($d \leq z \leq 1$) eq. (2.1) may be written

$$\frac{d^2 W}{d\xi^2} - \left\{ 1 - \frac{J_2 (1-d)^2 / \cos^2\theta}{\xi^2} \right\} W = 0 \qquad (2.4)$$

where $\xi = k \left\{ z - 1 + (1 - d)\, c \right\}$

Solutions of eq. (2.4), W_3 and W_4, are of the form $\xi^{1/2}\, Zm\,(i\,\xi)$ where

$$m^2 = \frac{1}{4} - \frac{J_2\,(1-d)^2}{\cos^2\theta}$$

2.3 The Static Stability

Three effects are present in the problem: shear, stratification, and the density discontinuity. Before attacking the general problem it will be useful to consider the two special cases in which the shear or the stratification is absent from the problem. In this section we consider the problem without shear, the static stability. If $U = 0$ eq (2.1) takes the simple form

$$\frac{d^2 W}{d z^2} - k^2 \left\{ 1 + J/\sigma^2 \right\} W = 0$$

where σ is the amplification factor

$$\sigma = i\,\alpha\, c$$

Assuming J_1 and J_2 to be constant, solutions are easily obtained, and, after imposing the boundary conditions, the following eigenvalue equation for σ is obtained:

$$\sqrt{1 + J_1/\sigma^2}\ \coth\ k d \sqrt{1 + J_1/\sigma^2}$$

$$+ \sqrt{1 + J_2/\sigma^2}\ \coth\ k\,(1-d) \sqrt{1 + J_2/\sigma^2} = \Delta\rho\, g\, k/\sigma^2$$

If we plot the left and right hand sides of this equation as functions of σ, assuming σ real, (Fig. 3) we see that there is at least one root for every value of k and therefore there is always at least one unstable mode (Re $\sigma \neq 0$).

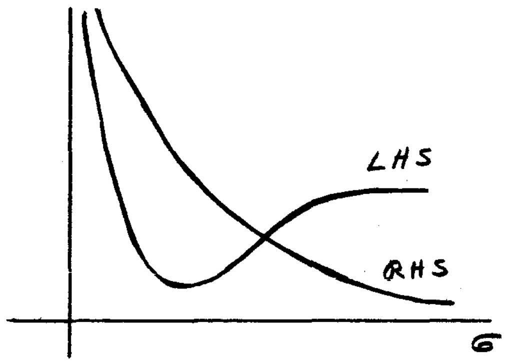

Fig. 3

Approximate solutions can be found for large and small values of the wavenumber:

A. Solution for large wavenumbers ($k >> 1$)

Assuming that σ^2 is of the order of magnitude of k, to first order we obtain

$$\sigma^2 = 1/2 \; \Delta \rho \; g k$$

This is identical with the result which is obtained when no stratification is present[3] (Taylor instability). This result is not surprising since we would not expect large wavenumbers (small eddies) to be sensitive to the structure of the stratification.

B. Solution for small wavenumbers ($k << d,\ 1-d$)

Assuming that σ is of the order of magnitude of k, to first order we obtain

$$\sqrt{J_1} \coth \frac{k d \sqrt{J_1}}{\sigma} + \sqrt{J_2} \coth \frac{k (1-d) \sqrt{J_2}}{\sigma} = \Delta \rho \, g \frac{k}{\sigma}$$

for $J_1, J_2 >> 1$ we have

$$\sigma \approx k \, \Delta \rho \, g / (\sqrt{J_1} + \sqrt{J_2})$$

Thus, the effect of the stratification is stabilizing with respect to small wavenumbers.

2.4 Taylor Stability with Shear

In this section we consider the second special case in which there is a jet profile and a density discontinuity without density stratification within the two layers (i.e., $J_1 = J_2 = 0$). In this case eq. (2.1) reduces to

$$\frac{d^2 w}{d z^2} - k^2 w = 0$$

After imposing the boundary conditions we obtain the eigenvalues:

$$1 - c = \frac{1}{2 k f(k)} \left\{ \delta \pm \sqrt{\delta^2 - 4 k f(k) \frac{\Delta \rho g}{\cos^2 \theta}} \right\}$$

where $f(k) = \coth k d + \coth k (1 - d)$

There is instability if the argument of the square root is negative. Depending on $\cos \theta$ we may have a cut-off wavenumber, k_c, with stability for $k < k_c$. Expanding $k f(k)$ in powers of k

$$k f(k) = \delta + 1/3\, k^2 + \ldots$$

We obtain k_c to first order by setting the argument of the square root ~~equal~~ to zero.

$$\delta^2 - 4 \frac{\Delta \rho g}{\cos^2 \theta} (\delta + 1/3\, k_c^2 + \ldots) = 0$$

$$k_c^2 \approx \frac{3}{4} \frac{\cos^2 \theta}{\Delta \rho g} \delta^2 - 3 \delta$$

Thus, there is a cut-off wavenumber provided

$$\cos^2 \theta > 4 \frac{\Delta \rho g}{\delta}$$

In other words, the cut-off exists for perturbations sufficiently close to the direction of the shear. In this sense, the effect of the shear is stabilizing.

3. Solution for Small Wavenumbers

We now attempt to obtain solutions of the general eigenvalue equation (2.1) in the approximation of small wavenumbers. J_1 and J_2 are assumed to be constant. As linearly independent solutions of Bessel's equation we choose J_n and J_{-n}. Thus,

$$W_1 = \varsigma^{1/2} J_n(i\varsigma) \qquad W_2 = \varsigma^{1/2} J_{-n}(i\varsigma)$$

$$W_3 = \xi^{1/2} J_m(i\xi) \qquad W_4 = \xi^{1/2} J_{-m}(i\xi)$$

Since $k \ll d$, $1-d$, we may expand the Bessel functions in power series. Substituting into eq. (2.1) we obtain, to first order

$$\frac{n}{d}\,\frac{z^{2n}+1}{z^{2n}-1} + \frac{m}{1-d}\,\frac{z^{2m}+1}{z^{2m}-1} = 1/2\,\delta - \frac{\Delta\rho\, g/\cos^2\theta}{1-c}$$

where $z = \varsigma(d)/\varsigma(0) = \xi(d)/\xi(1) = 1 - \frac{1}{c}$

This may be further simplified by letting $n = m$, i.e., $J_1\, d^2 = J_2\,(1-d)^2$. Then

$$n\,\frac{z^{2n}+1}{z^{2n}-1} = 1/2 - \frac{\varepsilon}{1-c} \tag{3.1}$$

where $\varepsilon = \Delta\rho\, g/\delta\cos^2\theta$

The effects of the density discontinuity and of the shear are represented by ε, while the effect of the stratification is represented by n. It will be convenient to make the following distinctions: weak Taylor instability ($\varepsilon \ll 1$), strong Taylor instability ($\varepsilon \gg 1$), weak stable stratification ($0 < n < 1/2$), strong stable stratification (n pure imaginary), and unstable stratification ($n > 1/2$). We shall first consider the case of weak Taylor instability.

3.1 Weak Taylor Instability ($\varepsilon \ll 1$)

3.11 Weak Stable Stratification ($0 < n < 1/2$)

In the "zeroth approximation", i.e., $\varepsilon = 0$, eq. (3.1) is

$$\frac{z^{2n} + 1}{z^{2n} - 1} = \frac{1}{2n} \tag{3.2}$$

which apparently has solutions

$$z = \left[\frac{1 + 2n}{1 - 2n}\right]^{\frac{1}{2n}} e^{\frac{l\pi i}{n}} \qquad l = \text{integer} \tag{3.3}$$

However, recalling that

$$z = 1 - \frac{1}{c} = 1 - \frac{c_r - i c_i}{|c|^2}$$

and assuming that $c_i > 0$, we may define the branch of z^{2n} by

$$0 < \arg z < \pi$$

and since in eq. (3.3) $\arg z = l\pi/n$ this implies that $0 < l < n$. But this is impossible since l is an integer and therefore in this case there are no amplified normal modes ($c_i > 0$).

There is a neutral mode (z real), however. This is easily seen by writing eq. (3.2) in the form

$$\coth n \log z = \frac{1}{2n}$$

Since $|\coth n \log z| > 1$, this equation has a root in the range $z > 1$ (or $c < 0$) (Fig. 4).

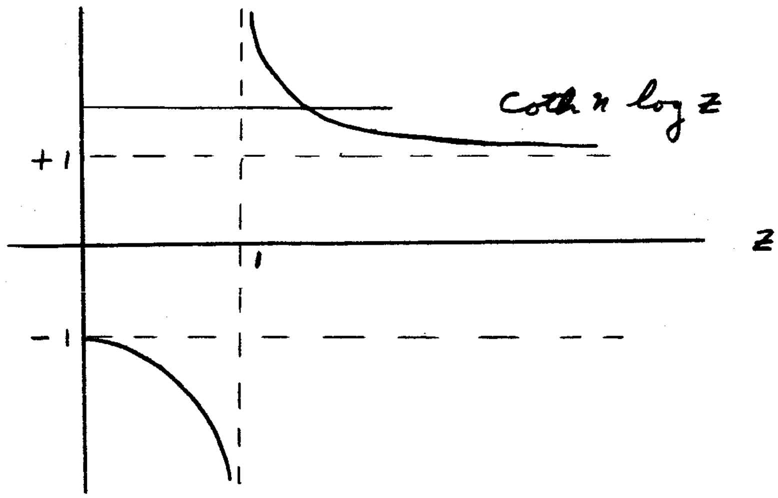

Fig. 4

It can be easily shown that there are no roots for $z < 0$.

We turn now to the case in which $\varepsilon \neq 0$. As a typical (and simple) example, let us set $n = 1/4$. Then eq. (3.1) may be written

$$x^3 \left\{ -1 + 4\varepsilon \right\} + x^2 \left\{ 3 - 4\varepsilon \right\} - 4\varepsilon x + 4\varepsilon = 0$$

where $x = z^{1/2}$. When $\varepsilon = 0$ we obtain the neutral mode already discussed, $x = 3$ ($c = -1/8$), in the zeroth approximation. We obtain this root in the first approximation by setting $x = 3 + x_1$, where $x_1 = \mathcal{O}(\varepsilon)$, and substituting into the cubic neglecting terms of order ε^2. Thus, we obtain

$$x = 3 + \frac{64}{9}\varepsilon + \mathcal{O}(\varepsilon^2)$$

The mode is still neutral in the first approximation. However, there are two other roots of the cubic which we may obtain in the first approximation by assuming $x = \mathcal{O}(\varepsilon^{1/2})$.

The result is (to second order)

$$x = \pm 2i\, \varepsilon^{1/2} / \sqrt{3} + \varepsilon/9 + \mathcal{O}(\varepsilon^{3/2})$$

or

$$c = 1 - \frac{4}{3}\varepsilon \left\{ 1 \pm i \frac{\varepsilon^{1/2}}{3^{3/2}} \right\} + \mathcal{O}(\varepsilon^2)$$

As a second example, when $n = 1/6$ the eigenvalue equation can be written as a quartic in x and in addition to the neutral mode we have three other modes:

$$x = (-1)^{1/3} \sqrt{\frac{3}{2}}\, \varepsilon^{1/3} \left\{ 1 + (-1)^{-2/3} \left(\frac{2}{3}\right)^{2/3} \varepsilon^{1/3} \right\} + \mathcal{O}(\varepsilon)$$

We find then that there is instability for any non-zero ε, however small. This is rather surprising if we recall the result of section 2.4. There we saw that without stratification ($J_1 = J_2 = 0$) there is cut-off wavenumber below which the system is stable if $\varepsilon < 1/4$. Here we see that there is no such stable regime so that the effect of a stable stratification is destabilizing with respect to the small wavenumbers.

3.12 Unstable Stratification

If $n > 1/2$ ($J_1, J_2 < 0$) solutions of the type (3.3) are possible. It is now convenient to write these solutions in the form

$$z = \left[\frac{2n+1}{2n-1}\right]^{\frac{1}{2n}} e^{\frac{p i \pi}{2n}}$$

or

$$c = \exp i \, a n \left\{ 1 \pm \left(\frac{2n+1}{2n-1}\right)^{\frac{1}{2n}} e^{-p\pi i/2n} \right\}$$

where p is an odd integer. Here arg z = $p\pi/2n$ and from the definition of the branch cut $0 < p < 2n$. Therefore, for $1/2 < n < 3/2$ there is one amplified mode, for for $3/2 < n < 5/2$ there are two amplified modes, etc.

In the next order of approximation there is a correction of order ε :

$$c = c_0 \left\{ 1 - \frac{\varepsilon}{\frac{1}{4} - n^2} + O(\varepsilon^2) \right\}$$

where c_o is the zeroth order root.

3.13 Strong Stable Stratification

In this case n is pure imaginary. Let $n = i\nu$. The eigenvalue equation may be written

$$\cot \nu \log z = \frac{1}{\nu} \left\{ \frac{1}{2} - \varepsilon + \frac{\varepsilon}{z} \right\} \qquad (3.4)$$

The nature of the neutral spectrum when $z > 0$ may be seen graphically (Fig. 5)

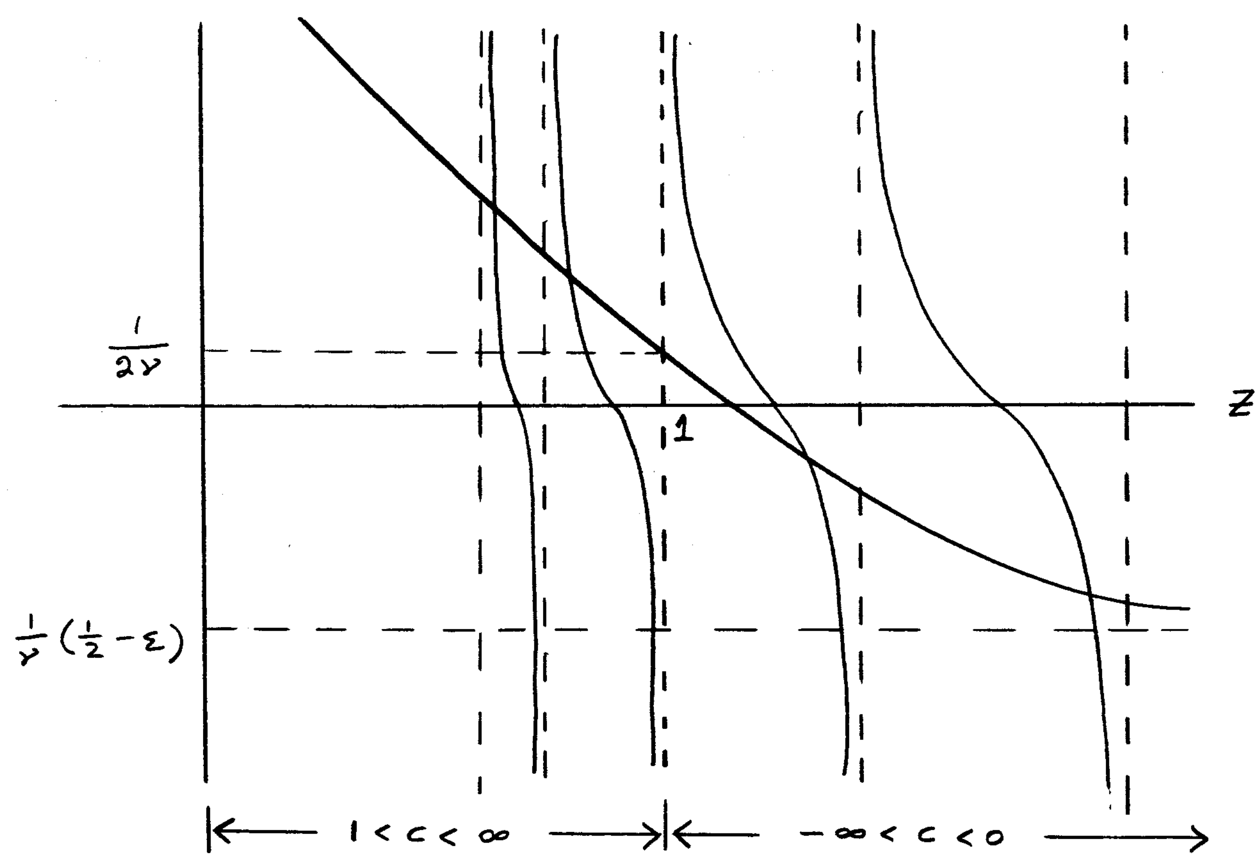

Fig. 5

There is an infinity of eigenvalues in the range $-\infty < c < 0$ and in the range $1 < c < \infty$ with points of accumulation at $c = 0, 1$. By separating eq. (3.4) into its real and imaginary parts it can be readily shown that there are no neutral modes in the range $0 < c < 1$ $(z < 0)$.

A theorem of Synge[4] may be extended to include the present problem, viz., for amplified modes the real part of c must lie in the range of the velocity profile. Here, this means that $0 < c_r < 1$.

We write eq. (2.1) in the form

$$[\rho (U-c)^2 F']' + \rho \left\{ \frac{J}{\cos^2\theta} - k^2 (U-c)^2 \right\} F = 0$$

where $W = (U-c)\ F$ and the Richardson number $J(z)$ includes a delta function because of the density discontinuity. We multiply by F^* and integrate over the range of U, (z_1, z_2). Assuming $F = 0$ on z_1 and z_2, after integrating the first term by parts we have

$$\int dz\, \rho (U-c)^2 Q - \int \frac{J}{\cos^2\theta} |F|^2 \rho\, dz = 0$$

where $Q = \rho \left\{ |F'|^2 + k^2 |F|^2 \right\}$

The imaginary part of this equation is

$$c_i \int dz\, (U - c_r)\, Q = 0$$

so that if $c_i \neq 0$, since Q is a positive function we must have $U_{min} < c_r < U_{max}$

Since in the present problem there are no neutral modes in this range except at the end points, any unstable modes must appear in a neighborhood of the points of accumulation, $c = 0, 1$.

3.2 Strong Taylor Instability

In this case $(\varepsilon >> 1)$ we would expect the density discontinuity to be the dominant effect. When only a density discontinuity is present the amplification factor, αc_i, is proportional to $k\sqrt{g\,\Delta\rho}$. It is therefore reasonable to assume that c is proportional to $\sqrt{\varepsilon}$. The assumption is, in fact, found to be justified a posteriori and in the first approximation $c = i\sqrt{\varepsilon}$. Further, c may be expressed as an expansion in powers of $\sqrt{\varepsilon}$. We expand z in powers of c to sufficient order and from eq. (3.1) obtain successive approximations to c. To the third order we obtain

$$c = i\sqrt{\varepsilon} + \frac{1}{2} + i\,\varepsilon^{-1/2}\left(\frac{2n^2+1}{12}\right) + \mathcal{O}(\varepsilon^{-1})$$

To this order the instability is an amplified wave with a wave velocity equal to the mean velocity of the shear profile. The effect of the stratification is relatively unimportant here, since it does not enter until the third approximation. It is interesting to note, however, that the effect is destabilizing even in the case of strong stable stratification unless $\nu > 1/\sqrt{2}$.

4. Solution for large Wavenumbers

When the wavenumber is large ($k >> 1$), we may use asymptotic solutions of eq. (2.3). As linearly independent solutions of Bessel's equation we now choose the Hankel functions of the first and second kind. Their asymptotic expansions are

$$H_n^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \left\{ 1 - \frac{4n^2-1}{4}\,\frac{1}{2iz} + \cdots \right\}$$

$$H_n^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{-i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \left\{ 1 + \frac{4n^2-1}{4}\,\frac{1}{2iz} + \cdots \right\}$$

Using these expansions we obtain the following expressions which we substitute into eq. (2.2)

$$\frac{W_1}{W_2} = e^{2i\left(is - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \left\{ 1 + \frac{4n^2-1}{4s} + \cdots \right\}$$

$$\frac{W_1'}{W_2'} = -\,e^{2i\left(is - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \left\{ 1 + \frac{4n^2-1}{4s} + \cdots \right\}$$

$$\frac{W_2'}{W_2} = k\left\{ 1 + \frac{4n^2-1}{8}\,\frac{1}{s^2} + \cdots \right\}$$

With similar expressions involving W_3 and W_4 (we still assume $n = m$). Now it may be seen that in eq. (2.2) higher order terms arising from W_1/W_2 and W_1'/W_2' are exponentially small and therefore negligible compared with terms arising from W_2'/W_2. To second order, then, we have

$$\Gamma = k\left\{1 + \frac{4n^2-1}{8}\frac{1}{[\varsigma(d)]^2}\right\} + k\left\{1 + \frac{4n^2-1}{8}\frac{1}{[\xi(d)]^2}\right\}$$

$$= 2k + \frac{1}{k(1-c)^2}\frac{4n^2-1}{8}\frac{1-2d+2d^2}{d^2(1-d)^2}$$

We recall that

$$\Gamma = \frac{\delta}{1-c} - \frac{\Delta\rho g/\cos^2\theta}{(1-c)^2}$$

Solving for the eigenvalue, c, we find that

$$c_r = 1 - \delta/4k + \mathcal{O}(1/k^2)$$

and for the amplification factor we obtain

$$\alpha c_i = \sqrt{\frac{k\Delta\rho g}{2}}\left\{1 - \frac{1}{16\Delta\rho g}\frac{1}{k}\left(\delta^2\cos^2\theta + J_1 d^2\varphi\right) + \mathcal{O}(k^{-2})\right\}$$

where $$\varphi = \frac{(1-d)^2 + d^2}{d^2(1-d)^2} > 0$$

Thus, we have an amplified wave traveling approximately with the velocity of the shear at the interface. To first order the amplification is independent of both the stratification and the shear, and is, in fact, just what we would have if only the density discontinuity were present. This is not surprising since we would expect small wavelengths to be sensitive primarily to conditions at the interface. Both the stratification and the shear appear in the second order term where the effect of each is stabilizing. Moreover, we see that the most stable perturbations are those in the direction of the mean velocity ($\theta = 0$). This is also quite reasonable since in this direction the inhibiting effect of the velocity profile (which is of the stable type) should be at its maximum.

5. Conclusions

We have seen that the dominant instability occurs for large values of the perturbation wavenumber (small wavelengths) in which case the amplification rate is proportional to $\sqrt{k}$. These modes are primarily associated with the density discontinuity at the interface. The effects of the velocity profile and the density stratification, while stabilizing, are of a smaller order of magnitude.

For small values of the wavenumber the instabilities are characterized by the Richardson number of the stratification and the dimensionless number $\varepsilon = \Delta\rho g / \zeta \cos^2\theta$. Recalling that ζ is the negative of the jump in the slope of the velocity profile at the interface, we see that ε is the ratio of the "Richardson number" of the density discontinuity to the square of the characteristic number of the projection of the velocity profile along the direction of the perturbation. From the form of ε, then, the effect of the mean flow, which we have assumed to be of the jet type, must always be stabilizing.

We have seen that in the absence of stratification if $\varepsilon < 1/4$ there is a cut-off wavenumber below which we have stability. However, when a statically stable density stratification is included in the problem there is instability for all wavenumbers so that with respect to small wavenumbers the effect of a stable stratification is destabilizing.

6. Acknowledgments

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